## Chapter review 5

1 $l_1: \mathbf{r} = 3\mathbf{i} + s\mathbf{j} - \mathbf{k}$ and $l_2: \mathbf{r} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\mathbf{a} = 3\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = \mathbf{j}$

$\mathbf{c} = 9\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\mathbf{a} - \mathbf{c} = 3\mathbf{i} - \mathbf{k} - (9\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -6\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} = \mathbf{i}(1-0) - \mathbf{j}(0-0) + \mathbf{k}(0-1) = \mathbf{i} - \mathbf{k}$$

Therefore:

$$\left| \frac{(-6\mathbf{i} + 2\mathbf{j}).(\mathbf{i} - \mathbf{k})}{|\mathbf{i} - \mathbf{k}|} \right| = \left| \frac{-6}{\sqrt{1^2 + (-1)^2}} \right| = \left| \frac{-6}{\sqrt{2}} \right| = 3\sqrt{2}$$

2 $l_1: \mathbf{r} = (3s - 3)\mathbf{i} - s\mathbf{j} + (s+1)\mathbf{k}$ and $l_2: \mathbf{r} = (3+t)\mathbf{i} + (2t-2)\mathbf{j} + \mathbf{k}$

$l_1: \mathbf{r} = -3\mathbf{i} + \mathbf{k} + s(3\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $l_2: \mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} + 2\mathbf{j})$

$\mathbf{a} = -3\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + 2\mathbf{j}$

$$\mathbf{a} - \mathbf{c} = -3\mathbf{i} + \mathbf{k} - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -6\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = \mathbf{i}(0-2) - \mathbf{j}(0-1) + \mathbf{k}(6+1) = -2\mathbf{i} + \mathbf{j} + 7\mathbf{k}$$

Therefore:

$$\left| \frac{(-6\mathbf{i} + 2\mathbf{j}).(-2\mathbf{i} + \mathbf{j} + 7\mathbf{k})}{|-2\mathbf{i} + \mathbf{j} + 7\mathbf{k}|} \right| = \left| \frac{12+2}{\sqrt{(-2)^2 + 1^2 + 7^2}} \right| = \left| \frac{14}{\sqrt{54}} \right| = \frac{7\sqrt{6}}{9}$$

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**3 a** $\overrightarrow{AB} = (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) - (-\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{CD} = (\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$p = \overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ -2 & 3 & -5 \end{vmatrix} = i - j - k$$

**b** $\overrightarrow{AC} = (2\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) - (-\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

$\overrightarrow{AC}.p = (2i - j + 5k).(i - j - k) = 2 + 1 - 5 = -2$

**c** The line containing $AB$ has equation $\mathbf{r} = -\mathbf{j} + 2\mathbf{k} + \lambda \overrightarrow{AB}$

The line containing $CD$ has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + \mu \overrightarrow{CD}$

So the shortest distance between the lines containing $AB$ and the line containing $CD$ is

$$\left| \frac{(-\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}).\overrightarrow{AB} \times \overrightarrow{CD}}{|\overrightarrow{AB} \times \overrightarrow{CD}|} \right| = \left| \frac{\overrightarrow{AC}.p}{|p|} \right| = \frac{2}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

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**4 a** Let $\mathbf{m} = \overrightarrow{OM} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
Then we seek **r** such that $\mathbf{r} \times \mathbf{m} = 5\mathbf{i} - 10\mathbf{k}$

Let **r** = (*a*, *b*, *c*) be any solution satisfying this equation.

$$\mathbf{r} \times \overrightarrow{OM} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ -4 & 1 & -2 \end{vmatrix} = \mathbf{i}(-2b - c) - \mathbf{j}(-2a + 4c) + \mathbf{k}(a + 4b)$$

So:
$$\mathbf{i}(-2b - c) - \mathbf{j}(-2a + 4c) + \mathbf{k}(a + 4b) = 5\mathbf{i} - 10\mathbf{k}$$
Hence:
$-2b - c = 5$ **(1)**
$-2a + 4c = 0$ **(2)**
$a + 4b = -10$ **(3)**
As the solution will be a line, any one of these letters can be arbitrary.
Try an arbitrary value $c = 1$:
Then from **(2)**: $-2a + 4 = 0$ so $a = 2$
Then from **(1)**: $-2b - 1 = 5$ so $b = -3$
Therefore **r** = (1, –3, 2) is on the line *l*.

Now note that as $\mathbf{m} \times \mathbf{m} = 0$, $(\mathbf{r} + t\mathbf{m}) \times \mathbf{m} = 5\mathbf{i} - 10\mathbf{k}$
So the equation of the line *l* is:
$$\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix}$$

**b** When $\lambda = 0$, $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ hence (2, −3, 1) lies on *l*.

$$\text{Area} = \frac{1}{2}\left|(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (-4\mathbf{i} + \mathbf{j} - 2\mathbf{k})\right|$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -4 & 1 & -2 \end{vmatrix} = \mathbf{i}(6 - 1) - \mathbf{j}(-4 + 4) + \mathbf{k}(2 - 12)$$
$$= 5\mathbf{i} - 10\mathbf{k}$$

$$\text{Area} = \frac{1}{2}|5\mathbf{i} - 10\mathbf{k}|$$
$$= \frac{1}{2}\sqrt{5^2 + (-10)^2}$$
$$= \frac{5\sqrt{5}}{2}$$

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**5 a** $l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $l_2 : \mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \mathbf{i}(2+3) - \mathbf{j}(1-6) + \mathbf{k}(-1-4)$$

$$= 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$$

**b** Since $\overrightarrow{AB}$ is perpendicular to $l_1$ and $l_2$ it is of the form $k\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Let $A$ be the point $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and let $B$ be the point $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$

Then:

$$\overrightarrow{AB} = \begin{pmatrix} d-a \\ e-b \\ f-c \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Since $A$ lies on $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1+\lambda \\ -1+2\lambda \\ 3\lambda \end{pmatrix}$$

Since $B$ lies on $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ 1-\mu \\ 1+\mu \end{pmatrix}$$

Hence:

$$\begin{pmatrix} 1+2\mu-\lambda \\ 2-\mu-2\lambda \\ 1+\mu-3\lambda \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$1+2\mu-\lambda = k$ **(1)**

$2-\mu-2\lambda = k$ **(2)**

$1+\mu-3\lambda = -k$ **(3)**

Adding **(2)** and **(3)** gives:

$$3 - 5\lambda = 0 \Rightarrow \lambda = \frac{3}{5}$$

subtracting **(2)** from **(1)** gives:

$$-1 + 3\mu + \lambda = 0$$

Substituting $\lambda = \frac{3}{5}$ gives:

$$-1 + 3\mu + \frac{3}{5} = 0 \Rightarrow \mu = \frac{2}{15}$$

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When $\lambda = \frac{3}{5}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{5} \\ \frac{1}{5} \\ \frac{9}{5} \end{pmatrix}$$

Hence $A$ is the point $\left(\frac{8}{5}, \frac{1}{5}, \frac{9}{5}\right)$

When $\mu = \frac{2}{15}$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{15}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{34}{15} \\ \frac{13}{15} \\ \frac{17}{15} \end{pmatrix}$$

Hence $B$ is the point $= \left(\frac{34}{15}, \frac{13}{15}, \frac{17}{15}\right)$

**6 a** $\overrightarrow{AB} = (3i + j + 4k) - (i + 3j + 3k) = 2i - 2j + k$

$\overrightarrow{AC} = (2i + 4j + k) - (i + 3j + 3k) = i + j - 2k$

A vector normal to the plane ABC is the direction $\overrightarrow{AB} \times \overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3i + 5j + 4k$$

A unit vector normal to the plane is $\frac{1}{\sqrt{3^2 + 5^2 + 4^2}}(3i + 5j + 4k) = \frac{1}{\sqrt{50}}(3i + 5j + 4k)$

**b** Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ and $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ (note **a** can be the position vector of any point on the plane), this gives a vector equation of the plane as:

$r.(3i + 5j + 4k) = (i + 3j + 3k).(3i + 5j + 4k) = 3 + 15 + 12 = 30$

So $3x + 5y + 4z = 30$ is a Cartesian equation of the plane.

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**6 c** The perpendicular distance from the origin to a plane with equation $\mathbf{r.n} = k$ where **n** is a unit vector perpendicular to the plane is $k$.

So from part **b**, the vector equation of the plane is $r.\frac{1}{\sqrt{50}}(3i+5j+4k)=\frac{30}{\sqrt{50}}$

So the perpendicular distance from the origin to the plane $=\frac{30}{\sqrt{50}}=\frac{30\sqrt{50}}{50}=3\sqrt{2}$

**7 a** Two non-parallel lines in the plane with vector equation $r=i+sj+t(i-k)$ are $j$ and $i-k$

So a normal to the plane is $j\times i-k=\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}=-i-k$

As $i+k$ is parallel to $-i-k$, it must be is perpendicular to the plane.

**b** From part **b**, $\mathbf{n}=\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ is a unit vector perpendicular to the plane.

Using $\mathbf{r.n}=\mathbf{a.n}$, with $\mathbf{a}=\mathbf{i}$, this gives a vector equation of the plane as

$r.\frac{1}{\sqrt{2}}(i+k)=(i).\frac{1}{\sqrt{2}}(i+k)=\frac{1}{\sqrt{2}}$

So as $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ is a unit vector,

the perpendicular distance from the origin to the plane $=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

**c** As $r.\frac{1}{\sqrt{2}}(i+k)=\frac{1}{\sqrt{2}}$ is a vector equation of the plane

A Cartesian equation of the plane is $\frac{1}{\sqrt{2}}(x+z)=\frac{1}{\sqrt{2}}$, which simplifies to $x+z=1$

**8 a** $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(5\mathbf{i}-2\mathbf{j}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})=4\mathbf{i}-3\mathbf{j}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})=2\mathbf{i}+\mathbf{j}+5\mathbf{k}$

A perpendicular vector to the plane is in direction $\overrightarrow{AB}\times\overrightarrow{AC}$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 0 \\ 2 & 1 & 5 \end{vmatrix}=-15\mathbf{i}-20\mathbf{j}+10\mathbf{k}$

**b** The equation of the plane containing $A$, $B$ and $C$ is

Using $\mathbf{r.n}=\mathbf{a.n}$, with $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, this gives a vector equation of the plane as

$r.(-15i-20j+10k)=(i+j+k).(-15i-20j+10k)=-15-20+10=-25$

So a Cartesian equation of the plane is

$-15x-20y+10z=-25$, which simplifies to $3x+4y-2z-5=0$

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**8 c** $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathrm{j} + \mathbf{k}) = 4\mathbf{j} + 5\mathbf{k}$

Volume of tetrahedron $ABCD = \frac{1}{6}\left|\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC})\right|$

$$= \frac{1}{6}\left|(4\mathbf{j} + 5\mathbf{k}).(-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k})\right| = \frac{1}{6}\left|(-80 + 50)\right| = \frac{30}{6} = 5$$

**9 a** $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2i - 3j) - (3i - 5j - k) = -i + 2j + k$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2i - 3j) - (-i + 5j + 7k) = 3i - 8j - 7k$

$$\overrightarrow{AC} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & -8 & -7 \end{vmatrix} = -6i - 4j + 2k$$

**b** $\overrightarrow{AB} \times \overrightarrow{AC}$ is a normal to the plane $\Pi$ and $3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ is a point on the plane

So an equation of the plane is

$r.(-6i - 4j + 2k) = (3i - 5j - k).(-6i - 4j + 2k) = -18 + 20 - 2 = 0$

This simplifies to $r.(3i + 2j - k) = 0$

**c** As 3**i** + 2**j** – **k** is a normal to the plane, the perpendicular from the point (2, 3, –2) to the plane has the equation

$r = 2i + 3j - 2k + \lambda(3i + 2j - k)$

Using the result from part **b**, this meets the plane when

$\left((2 + 3\lambda)i + (3 + 2\lambda)j + (-2 - \lambda)k\right).(3i + 2j - k) = 0$

$\Rightarrow 3(2 + 3\lambda) + 2(3 + 2\lambda) - 1(-2 - \lambda) = 0$

$\Rightarrow 14\lambda + 14 = 0$

$\Rightarrow \lambda = -1$

Substitute $\lambda = -1$ into the equation of the line gives

$r = 2i + 3j - 2k + (-1)(3i + 2j - k) = -i + j - k$

So the perpendicular from (2, 3, –2) meets the plane at (–1, 1, –1)

**10 a** $$p \times q = (3i - j + 2k) \times (2i + j - k) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -i + 7j + 5k$$

**b** $\mathbf{p} \times \mathbf{q}$ is a normal to the plane and the point with position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is on the plane, so an equation of the plane is

$r.(-i + 7j + 5k) = (3i - j + 2k).(-i + 7j + 5k) = -3 - 7 + 10 = 0$

So a Cartesian equation for the plane is $-x + 7y + 5z = 0$

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**10 c** $(r-p)\times q=0$ is one form of the vector equation of a line passing through the point with position vector **p** and parallel to the vector **q**. So the equation can also be written as
$r=pq+\lambda q$, i.e. $r=3i-j+2k+\lambda(2i+j-k)$
This meets the plane $r.(i+j+k)=2$ when
$((3+2\lambda)+(-1+\lambda)+(2-\lambda)).(\mathbf{i}+\mathbf{j}+\mathbf{k})=2$
$\Rightarrow(3+2\lambda)+(-1+\lambda)+(2-\lambda)=2\Rightarrow 2\lambda+4=2\Rightarrow\lambda=-1$
Substitute $\lambda=-1$ into the equation of the line gives
$r=3i-j+2k+(-1)(2i+j-k)=i-2j+3k$
So the coordinates of point $T$ are (1, –2, 3)

**11 a** Let the respective normal to each plane be $\mathbf{n}_1$ and $\mathbf{n}_2$, then
$n_1=2i+2j-k$ and $n_2=i-2j$
Let the acute angle between the two planes be $\theta$, then $\theta$ is also the angle between the respective normal to each plane, so

$$\cos\theta=\left|\frac{n_1.n_2}{|n_1||n_2|}\right|=\frac{|2\times1-2\times2|}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+(-2)^2}}=\frac{2}{3\sqrt{5}}=\frac{2\sqrt{5}}{15}$$

$\Rightarrow\theta=72.7°=73°$ (to the nearest degree)

**b** The direction of the line of intersection is perpendicular to the normal of each plane.

Hence the direction is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&-1\\1&-2&0\end{vmatrix}=-2\mathbf{i}-\mathbf{j}-6\mathbf{k}$

Any scalar multiple of this vector is also in the direction of the line of intersection, so simplify by multiplying by –1 to get $2\mathbf{i}+\mathbf{j}+6\mathbf{k}$

Find a point on the line by setting $y=0$ and solving the Cartesian equations of the two planes.

$2x+2y-z=9$ **(1)**
$x-2y=7$ **(2)**
Substituting for $y$ in equation **(2)** gives: $x=7$
Substituting for $x$ and $y$ in equation **(1)** gives: $2\times7-z=9\Rightarrow z=5$
So $7\mathbf{i}+5\mathbf{k}$ is the position vector of a point on the line of intersection

A line passing through a point with position vector **a** and parallel to vector **b** has the vector equation $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$, so an equation of the line of intersection is

$$\mathbf{r}\times(2\mathbf{i}+\mathbf{j}+6\mathbf{k})=(7\mathbf{i}+5\mathbf{k})\times(2\mathbf{i}+\mathbf{j}+6\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\7&0&5\\2&1&6\end{vmatrix}=-5\mathbf{i}-32\mathbf{j}+7\mathbf{k}$$

So the equation is $\mathbf{r}\times(2\mathbf{i}+\mathbf{j}+6\mathbf{k})=-5\mathbf{i}-32\mathbf{j}+7\mathbf{k}$

**12 a** $\overrightarrow{PS}=\overrightarrow{OS}-\overrightarrow{OP}$
$=\mathbf{i}+\mathbf{j}+4\mathbf{k}-2\mathbf{i}$
$=-\mathbf{i}+\mathbf{j}+4\mathbf{k}$

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**12 b** $\begin{pmatrix}1\\1\\4\end{pmatrix}.\begin{pmatrix}0\\-4\\1\end{pmatrix}=0-4+4=0$

therefore, $\overrightarrow{OS}$ and $=-4\mathbf{j}+\mathbf{k}$ are perpendicular

$\begin{pmatrix}-1\\1\\4\end{pmatrix}.\begin{pmatrix}0\\-4\\1\end{pmatrix}=0-4+4=0$

therefore, $\overrightarrow{PS}$ and $=-4\mathbf{j}+\mathbf{k}$ are perpendicular

**c** $\overrightarrow{SQ}=\overrightarrow{OQ}-\overrightarrow{OS}$

$=2\mathbf{i}+2\mathbf{j}-(\mathbf{i}+\mathbf{j}+4\mathbf{k})$

$=\mathbf{i}+\mathbf{j}-4\mathbf{k}$

As $-4\mathbf{j}+\mathbf{k}$ is normal to the plane $OSP$,

$$\sin\theta=\frac{(\mathbf{i}+\mathbf{j}-4\mathbf{k}).(-4\mathbf{j}+\mathbf{k})}{|\mathbf{i}+\mathbf{j}-4\mathbf{k}|\times|-4\mathbf{j}+\mathbf{k}|}$$

$$=\frac{-4-4}{\sqrt{1^2+1^2+(-4)^2}\times\sqrt{(-4)^2+1^2}}$$

$$=\frac{-8}{\sqrt{18}\times\sqrt{17}}$$

$\theta=-27.21...$

Therefore, the acute angle is 27° (to the nearest degree)

**13 a** The normal to the plane $\Pi$ is in the direction

$$(4i+j+2k)\times(3i+2j-k)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&1&2\\3&2&-1\end{vmatrix}=-5i+10j+5k$$

The line $L$ is in the direction $(2i+3j-4k)$

Finding the scalar product of the direction of the normal to the plane and the direction of the line

$(-5i+10j+5k).(2i+3j-4k)=-10+30-20=0$

This means that the line $L$ is perpendicular to the normal to the plane, so the line $L$ is parallel to the plane $\Pi$.

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**13 b** The line $L$ passes through point (2, 1, –3)

The perpendicular to plane $\Pi$ through the point (2, 1, –3) has a vector equation

$r = 2i + j - 3k + \lambda(-5i + 10j + 5k)$

As the normal to the plane is $-5i + 10j + 5k$ and $i + 3j + 4k$ is the position vector of a point on the plane, the equation of the plane may be written as

$r.(-5i + 10j + 5k) = (i + 3j + 4k).(-5i + 10j + 5k) = -5 + 30 + 20 = 45$

So the perpendicular to the plane $\Pi$ from (2, 1, –3) meets the plane when

$\left((2 - 5\lambda)i + (1 + 10\lambda)j + (-3 + 5\lambda)k\right).(-5 + 10j + 5k) = 45$

$\Rightarrow -10 + 25\lambda + 10 + 100\lambda - 15 + 25\lambda = 45$

$\Rightarrow 150\lambda = 60 \Rightarrow \lambda = \frac{2}{5}$

Substituting $\lambda = \frac{2}{5}$ into the equation of the perpendicular to plane $\Pi$ through the point (2, 1, –3) gives $r = 5j - k$, so the perpendicular to $\Pi$ from (2, 1, –3) meets the plane at (0, 5, –1). As the line is parallel to the plane, the shortest distance from $L$ to $\Pi$ is the distance between these points, i.e.

$$\sqrt{(2-0)^2 + (1-5)^2 + (-3-(-1))^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

Alternatively, note that as $L$ is parallel to the plane $\Pi$, the shortest distance between $L$ and the plane will also be the shortest distance between $L$ and any line $L_1$ on the plane that is non-parallel with $L$. These two lines are skew.

Write the equation of $L$ as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

And $L_1$ as $\mathbf{r} = \mathbf{c} + s\mathbf{d}$, where $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, a point on the plane, and $\mathbf{d} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, a direction on the plane

$$b \times d = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & 1 & 2 \end{vmatrix} = 10i - 20j - 10k$$

Using the result for the shortest distance between two skew lines

$$\text{Shortest distance} = \left|\frac{(\mathbf{a}-\mathbf{c}).(\mathbf{b}\times\mathbf{d})}{|(\mathbf{b}\times\mathbf{d})|}\right| = \left|\frac{(\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}).(10\mathbf{i} - 20\mathbf{j} - 10\mathbf{k})}{\sqrt{10^2 + (-20)^2 + (-10)^2}}\right|$$

$$= \frac{10 + 40 + 70}{\sqrt{600}} = \frac{120}{10\sqrt{6}} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

**14 a** $\Pi_1 : \mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$, $\Pi_2 : \mathbf{r}\cdot(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$ and $A$ is the point (2, −2, 3)

$\Pi_2 : x + 5y + 3k = 1$

Substituting (2, −2, 3) gives:

$2 + 5(-2) + 3(3) = 1$

$2 - 10 + 9 = 1$

$1 = 1$

Therefore, (2, −2, 3) lies on $\Pi_2$

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**14 b** $\begin{pmatrix}2\\-1\\1\end{pmatrix}.\begin{pmatrix}1\\5\\3\end{pmatrix}=2-5+3=0$

therefore, the planes are perpendicular

**c** $\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$

**d** $\mathbf{r}=\begin{pmatrix}2+2\lambda\\-2-\lambda\\3+\lambda\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$=2(2+2\lambda)-1(-2-\lambda)+1(3+\lambda)$

$=6\lambda+9$

$6\lambda+9=0\Rightarrow\lambda=-\frac{3}{2}$

Substituting $\lambda=-\frac{3}{2}$ into $\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ gives:

$\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}-\frac{3}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix}$

$=\begin{pmatrix}-1\\-\frac{1}{2}\\\frac{3}{2}\end{pmatrix}$

Therefore, they meet at the point $\left(-1,-\frac{1}{2},\frac{3}{2}\right)$

**e** The unit vector parallel to

$2\mathbf{i}-\mathbf{j}+\mathbf{k}=\frac{1}{|2\mathbf{i}-\mathbf{j}+\mathbf{k}|}(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

$=\frac{1}{\sqrt{6}}(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

The plane $\Pi'$ passing through (2, −2, 3) has equation:

$\mathbf{r}\cdot\frac{1}{\sqrt{6}}(2\mathbf{i}-\mathbf{j}+\mathbf{k})=(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})\frac{1}{\sqrt{6}}(2\mathbf{i}-\mathbf{j}+\mathbf{k})$

$=\frac{1}{\sqrt{6}}(4+2+3)$

$=\frac{3\sqrt{6}}{2}$

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**14 f** $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+\mathbf{k})=(2\mathbf{i}-2\mathbf{j}+3\mathbf{k}).(2\mathbf{i}-\mathbf{j}+\mathbf{k})$
$=4+2+3$
$\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=9$

**15 a** A normal to the plane is $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ so the line $l$ is parallel to this vector and it passes through the point with position vector $\mathbf{i}+2\mathbf{j}+\mathbf{k}$, hence a vector equation of the line is
$r=i+2j+k+\lambda(2i+j+3k)$

**b** From the vector equation, the coordinates of a point on $l$ are $(1+2\lambda,\ 2+\lambda,\ 1+3\lambda)$
So the line $l$ meets the plane $\Pi$ when
$2(1+2\lambda)+(2+\lambda)+3(1+3\lambda)=21$
$\Rightarrow 14\lambda+7=21\Rightarrow\lambda=1$
Substitute $\lambda=1$ into the equation of the line $l$ gives $r=3i+3j+4k$
So $M$ has coordinates (3, 3, 4)

**c** $\overrightarrow{OP}\times\overrightarrow{OM}=(i+2j+k)\times(3i+3j+4k)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 3 & 3 & 4\end{vmatrix}=5i-j-3k$

**d** Let $\theta$ be the acute angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OM}$
Then, by simple geometry, the distance $d$ from $P$ to the line $OM$ is $|\overrightarrow{OP}|\sin\theta$

From the definition of the vector product $\sin\theta=\dfrac{|\overrightarrow{OP}\times\overrightarrow{OM}|}{|\overrightarrow{OP}||\overrightarrow{OM}|}$

So $d=|\overrightarrow{OP}|\sin\theta=\dfrac{|\overrightarrow{OP}||\overrightarrow{OP}\times\overrightarrow{OM}|}{|\overrightarrow{OP}||\overrightarrow{OM}|}=\dfrac{|\overrightarrow{OP}\times\overrightarrow{OM}|}{|\overrightarrow{OM}|}$

$=\dfrac{\sqrt{5^2+(-1)^2+(-3)^2}}{\sqrt{3^2+3^2+4^2}}=\dfrac{\sqrt{35}}{\sqrt{34}}$

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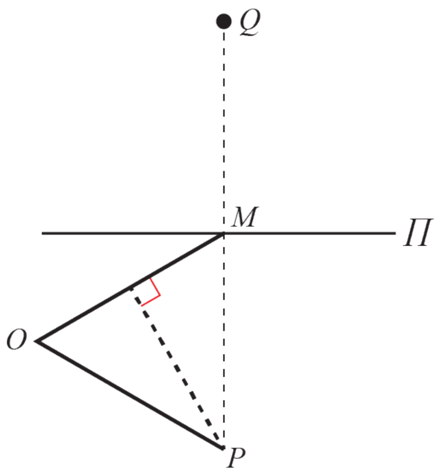
**15 e** This sketch shows the problem




$\overrightarrow{PM} = (3\mathbf{i}+3\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

Therefore $\overrightarrow{MQ} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

And $\overrightarrow{OQ} = \overrightarrow{OM}+\overrightarrow{MQ} = (3\mathbf{i}+3\mathbf{j}+4\mathbf{k})+(2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = 5\mathbf{i}+4\mathbf{j}+7\mathbf{k}$

So $Q$ has coordinates (5, 4, 7)

**16 a** $l_1 : \mathbf{r} = \mathbf{i}-\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$ and $l_2 : \mathbf{r} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}+\mu(-3\mathbf{i}+4\mathbf{k})$

When the lines intersect:

$$\begin{pmatrix}1+2\lambda\\-1+\lambda\\-2\lambda\end{pmatrix} = \begin{pmatrix}1-3\mu\\2\\2+4\mu\end{pmatrix}$$

$-1+\lambda = 2 \Rightarrow \lambda = 3$

$1+2(3) = 1-3\mu \Rightarrow \mu = -2$

$$\begin{pmatrix}1+2(3)\\-1+3\\-2(3)\end{pmatrix} = \begin{pmatrix}1-3(-2)\\2\\2+4(-2)\end{pmatrix}$$

$$\begin{pmatrix}7\\2\\-6\end{pmatrix} = \begin{pmatrix}7\\2\\-6\end{pmatrix}$$

Therefore, the lines intersect

**b** From part **a**

$\mathbf{r} = 7\mathbf{i}+2\mathbf{j}-6\mathbf{k}$

**16 c** $\cos\theta = \frac{(2\mathbf{i}+\mathbf{j}-2\mathbf{k})(-3\mathbf{i}+4\mathbf{k})}{|2\mathbf{i}+\mathbf{j}-2\mathbf{k}|\times|-3\mathbf{i}+4\mathbf{k}|}$

$$= \frac{-6-8}{\sqrt{2^2+1^2+(-2)^2}\times\sqrt{(-3)^2+4^2}}$$

$$= \frac{-14}{15}$$

$\theta = 158.9...$

Therefore, the acute angle is 21.03… and $\cos\theta = \frac{14}{15}$

**d** The vector equation of the plane will be of the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{v} + \mu\mathbf{w}$ where **a** lies on the plane, and **v** and **w** are vectors within it.

Take $\mathbf{a} = \mathbf{i} - \mathbf{j}$ from the equation of $l_1$
Take $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ also from the equation of $l_1$
Take $\mathbf{w} = -3\mathbf{i} + 4\mathbf{k}$ from the equation of $l_2$

Then a vector equation of the line is $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(-3\mathbf{i} + 4\mathbf{k})$

**17** Let the position vector of point $C$ relative to the origin be $\mathbf{c} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Then the volume of the tetrahedron is given by $\frac{1}{6}|\mathbf{c}.(\mathbf{a}\times\mathbf{b})|$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 0 \\ 2 & -1 & -3 \end{vmatrix} = -6\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}$$

This gives

$$\frac{1}{6}|\mathbf{c}.(\mathbf{a}\times\mathbf{b})| = \frac{1}{6}|(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(-6\mathbf{i}+15\mathbf{j}-9\mathbf{k})| = \frac{1}{6}|-6x+15y-9z| = \frac{1}{2}|-2x+5y-3z|$$

So if the volume is 5 $m^3$, then the locus of admissible points is

$$\frac{1}{2}|-2x+5y-3z| = 5 \Rightarrow |-2x+5y-3z| = 10$$

So Cartesian equations satisfying this equation are

$-2x+5y-3z = 10 \Rightarrow 2x-5y+3x+10 = 0$

and $2x-5y+3z = 10 \Rightarrow 2x-5y+3x-10 = 0$

**18 a** Equation of $L_1$ is $\mathbf{r} = 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + s(\mathbf{j} + 2\mathbf{k})$

When $s = 2$, $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ , so $P$ lies on $L_1$

Equation of $L_2$ is $\mathbf{r} = 8\mathbf{i} + 3\mathbf{j} + t(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

When $t = -1$, $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, so $P$ lies on $L_2$

**b** $$b_1 \times b_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10i + 10j - 5k$$

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**18 c** The normal to the plane is in direction of $b_1 \times b_2$. So $-2i + 2j - k$ is a normal to the plane.
Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{a} = 3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$ (note **a** can be the position vector of any point on the plane), this gives a vector equation of the plane as:
$\mathbf{r}.(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}).(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -6 - 6 + 2 = -10$
So $2x - 2y + z = 10$ is a Cartesian equation of the plane.

**d** $\overrightarrow{A_1P} = (3i - j + 2k) - (3i - 3j - 2k) = 2j + 4k = 2b_1$
$\overrightarrow{A_2P} = (3i - j + 2k) - (8i + 3j) = (-5i - 4j + 2k) = -b_2$

$$\text{Area of triangle } PA_1A_2 = \frac{1}{2}\left|\overrightarrow{A_1P} \times \overrightarrow{A_2P}\right| = \frac{1}{2}\left|2\mathbf{b}_1 \times -\mathbf{b}_2\right|$$
$$= \left|\mathbf{b}_1 \times \mathbf{b}_2\right| = \left|-10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}\right| \qquad \text{from part } \mathbf{b}$$
$$= \sqrt{(-10)^2 + (10)^2 + (-5)^2}$$
$$= \sqrt{225} = 15$$

**19 a** $A: a(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$, $B: a(-4\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and $C: a(5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k})$
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$$= a\begin{pmatrix} 5 \\ -2 \\ 11 \end{pmatrix} - a\begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$$

$$= a\begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}$$
Therefore:
$$\mathbf{r} = a\begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix} + \lambda a\begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}$$

**b** $OAB$ contains $O$:(0, 0, 0), $A$: $a$(5, −1, −3) and $B$: $a$(−4, 4, −1)
Hence:
$$\mathbf{r} = a\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} + \lambda a\begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} + \mu a\begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}$$

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**19 c** $\cos\theta = \frac{(5\mathbf{i}-\mathbf{j}-3\mathbf{k}).(-4\mathbf{i}+4\mathbf{j}-\mathbf{k})}{|5\mathbf{i}-\mathbf{j}-3\mathbf{k}|\times|-4\mathbf{i}+4\mathbf{j}-\mathbf{k}|}$

$= \frac{-20-4+3}{\sqrt{5^2+(-1)^2+(-3)^2}\times\sqrt{(-4)^2+4^2+(-1)^2}}$

$= \frac{-21}{\sqrt{35}\sqrt{33}}$

$\theta = 128.16...$

Therefore, the acute angle is 51.83… and $\cos\theta = \frac{21}{\sqrt{35}\sqrt{33}}$

**d** $\overrightarrow{BC}: \mathbf{r} = a\begin{pmatrix}-4\\4\\-1\end{pmatrix} + \lambda a\begin{pmatrix}9\\-6\\12\end{pmatrix}$ and $A$ is the point $a(5, -1, -3)$

$\mathbf{r}\cdot a\begin{pmatrix}9\\-6\\12\end{pmatrix} = a\begin{pmatrix}5\\-1\\-3\end{pmatrix} a\begin{pmatrix}9\\-6\\12\end{pmatrix}$

$a(9x-6y+12z) = a^2(45+6-36)$

$9x-6y+12z = 15a$

$3x-2y+4z = 5a$

**e** $\overrightarrow{BC}: \mathbf{r} = a\begin{pmatrix}-4\\4\\-1\end{pmatrix} + \lambda a\begin{pmatrix}9\\-6\\12\end{pmatrix}$

Written in Cartesian form this is:

$\frac{x+4a}{9} = \frac{y-4a}{-6} = \frac{z+a}{12} = \lambda$

**20 a** $\overrightarrow{BC} = (2i+3j+3k)-(i+2j+3k) = i+j$

$\overrightarrow{BD} = (3i+2j+4k)-(i+2j+3k) = 2i+k$

So $\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 1 & 0\\2 & 0 & 1\end{vmatrix} = i-j-2k$ which is normal to the plane $BCD$

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, this gives a vector equation of the plane $BCD$ as

$r.(i-j-2k) = (i+2j+3k).(i-j-2k) = 1-2-6 = -7$

This may be written in Cartesian form as $x-y-2z+7=0$

**b** Let $\alpha$ be the angle between $BC$ and the plane $x+2y+3z=4$ and $\theta$ be the acute angle between $BC$ and the normal to this plane, which is $i+2j+3k$.

Then $\alpha = 90-\theta \Rightarrow \sin\alpha = \cos\theta$

So $\sin\alpha = \cos\theta = \frac{|(i+j).(i+2j+3k)|}{\sqrt{1^2+1^2}\sqrt{1^2+2^2+3^2}} = \frac{3}{\sqrt{2}\sqrt{14}} = 0.567$ (3 s.f.)

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**20 c** Let $A$ have coordinates $(x, y, z)$

Then $\overrightarrow{AC} = (2-x)\mathbf{i} + (3-y)\mathbf{j} + (3-z)\mathbf{k}$ and $\overrightarrow{AD} = (3-x)\mathbf{i} + (2-y)\mathbf{j} + (4-z)\mathbf{k}$

As $AC$ is perpendicular to $BD$, $\overrightarrow{AC}.\overrightarrow{BD} = 0$

So $2(2-x) + (3-z) = 0$

$\Rightarrow 2x + z = 7$ **(1)**

As $AD$ is perpendicular to $BC$, $\overrightarrow{AD}.\overrightarrow{BC} = 0$

So $(3-x) + (2-y) = 0$

$\Rightarrow x + y = 5$ **(2)**

As $AB = \sqrt{26}$

$(x-1)^2 + (y-2)^2 + (z-3)^2 = 26$ **(3)**

Substituting $z = 7 - 2x$ and $y = 5 - x$ from equation **(1)** and **(2)** into equation **(3)** gives

$(x-1)^2 + (3-x)^2 + (4-2x)^2 = 26$

$x^2 - 2x + 1 + 9 - 6x + x^2 + 16 - 16x + 4x^2 = 26$

$6x^2 - 24x = 0$

$x(x-4) = 0$

$\Rightarrow x = 0$ or $4$

When $x = 0$, $y = 5$ and $z = 7$

When $x = 4$, $y = 1$ and $z = -1$

The two possible positions are (0, 5, 7) and (4, 1, –1)

**Challenge**

Two direction vectors in the plane given by $\mathbf{r}_1 = p\mathbf{i} - r\mathbf{k}$ and $\mathbf{r}_2 = q\mathbf{j} - r\mathbf{k}$

Hence a normal to the plane is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 0 & -r \\ 0 & q & -r \end{vmatrix} = qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}$$

Using $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, with $\mathbf{n} = qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}$ and $\mathbf{a} = p\mathbf{i}$, a point on the plane, this gives a vector equation of the plane as:

$$\mathbf{r}.qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k} = p\mathbf{i}.(qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}) = pqr$$

If $d$ is the length of the perpendicular from the origin to the plane then $\mathbf{r}.\dfrac{1}{|\mathbf{n}|}\mathbf{n} = d$

$$\text{So } d = \frac{pqr}{\sqrt{q^2r^2 + p^2r^2 + p^2q^2}}$$

$$\Rightarrow d^2 = \frac{p^2q^2r^2}{q^2r^2 + p^2r^2 + p^2q^2}$$

$$\Rightarrow \frac{1}{d^2} = \frac{q^2r^2 + p^2r^2 + p^2q^2}{p^2q^2r^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$$

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